Lecture 12

Tuesday, February 1, 2022 10:08 PM

* Prayer

* Spiritual thought

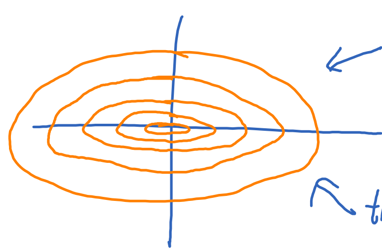
multivariable function $f(x,y,\dots)$ $\begin{cases} \text{graph} \\ \text{domain} \\ \text{level set} \end{cases}$

$f(x,y)$ $\begin{cases} \text{graph}: \{(x,y,z) : z = f(x,y)\} \\ \text{domain}: \{(x,y) : f(x,y) \text{ exists}\} \\ \text{level set } \{(x,y) : f(x,y) = \text{const}\} \\ \text{contour map}: \text{many level sets} \end{cases}$

Ex $f(x,y) = \dfrac{1}{x^2 + 4y^2}$

Domain $= \{(x,y) \mid (x,y) \neq (0,0)\}$

Level set $= \{(x,y) \mid x^2 + 4y^2 = \text{const}\}$



contour map

the value get higher and higher as the ring goes into the origin.

Calculus I $\begin{cases} \text{limit} \\ \text{derivative} \\ \text{integral} \end{cases}$ Calc. of Several Variables $\begin{cases} \text{limit} \\ \text{derivative} \\ \text{integral} \end{cases}$

* **Limit** $\lim_{(x,y,\dots)\to(x_0,y_0,\dots)} f(x,y,\dots) = L$ means:

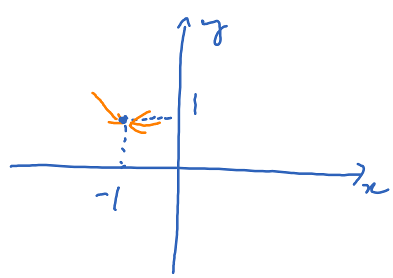
$f(x,y,\dots)$ gets arbitrarily close to $(x_0,y_0,\dots)$ if $(x,y,\dots)$ get close enough to $(x_0,y_0,\dots)$.

Ex $\lim_{(x,y)\to(-1,2)} f(x,y) = 1$ means:

$f(x,y)$ can be as close to 1 as one wishes given that (x,y) is close enough to $(-1,2)$.

Ex $\displaystyle\lim_{(x,y)\to(1,0)} \frac{x}{x^2+y^2+1} = \frac{1}{1+0+1} = \frac{1}{2}$.

Ex $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{x+y}{x-y+2}$



Test a few paths:

① $\begin{cases} x = -1+t \\ y = 1-t \end{cases}$

$$\frac{x+y}{x-y+2} = \frac{(-1+t)+(1-t)}{(-1+t)-(1-t)+2} = \frac{0}{2t} = 0 \xrightarrow{t\to 0} 0$$

② $\begin{cases} x = -1+t \\ y = 1 \end{cases}$

$$\frac{x+y}{x-y+2} = \frac{(-1+t)+1}{(-1+t)-1+2} = \frac{t}{t} = 1 \xrightarrow{t\to 0} 1$$

Two limits on two different paths are different.

$$\longrightarrow \lim_{(x,y)\to(-1,1)} \frac{x+y}{x-y+2} \quad \text{doesn't exist}$$

Ex: $$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim \frac{x^3}{x^2+y^2} - \lim \frac{y^3}{x^2+y^2}$$

Sandwich rule:

$$\left| \frac{x^3}{x^2+y^2} - 0 \right| = \frac{|x^3|}{x^2+y^2} = \frac{x^2}{x^2+y^2}|x| \leq |x| \longrightarrow 0$$

$$\left| \frac{y^3}{x^2+y^2} - 0 \right| = \frac{|y^3|}{x^2+y^2} = \frac{y^2}{x^2+y^2}|y| \leq |y| \longrightarrow 0$$

Hence,

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0 - 0 = 0$$

Mathematica:

$$\text{Limit}\left[\frac{x^3-y^3}{x^2+y^2}, \{x,y\} \to \{0,0\} \right]$$